Graph Theory – Part I

Aleksandra Gorzkowska

Definition

A (simple) graph is a pair (V, E), where V is a nonempty finite set and E is a subset of the set of two-element subsets of V.
Notation: $E \subset \binom{V}{2}$;
$\binom{V}{2}$ – the set of all two-element subsets of V

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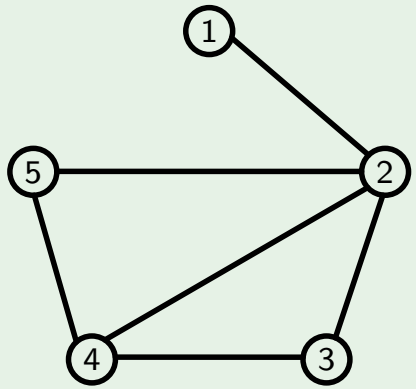
Terminology and notation:
The set V is called the vertex set of the graph;
the set E is called the edge set of the graph.

Notation and interpretation:
$x, y \in V$ – x and y are vertices of the graph;
$e = \{x, y\} \in E$ – the set $\{x, y\}$ is an edge of the graph (denoted by e);
for simplicity: $xy \in E$ – xy is an edge of the graph

Example



$V = \{1, 2, 3, 4, 5\}$
$E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}$
or simply: $E = \{12, 23, 24, 25, 34, 45\}$

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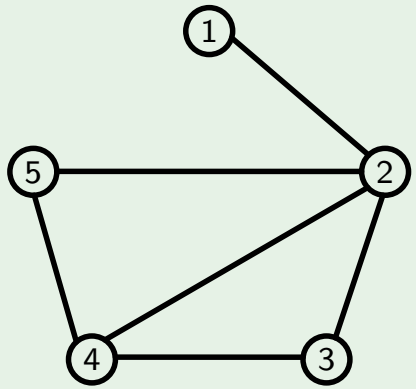

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The order of a graph $G = (V, E)$ is the number of its vertices, i.e. the cardinality of V.
Notation: $|G| := |V|$

The size of a graph $G = (V, E)$ is the number of its edges, i.e. the cardinality of E.
Notation: $||G|| := |E|$

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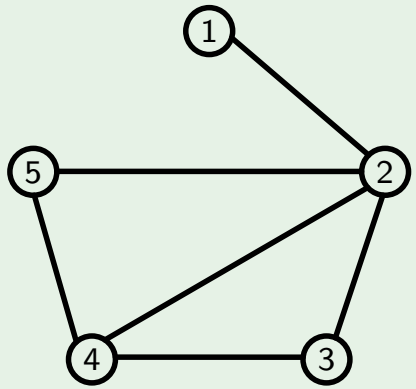

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or simply: $E = \{12, 23, 24, 25, 34, 45\}$
$|G| = 5$; $||G|| = 6$

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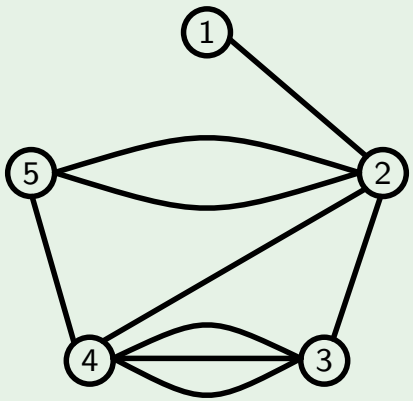
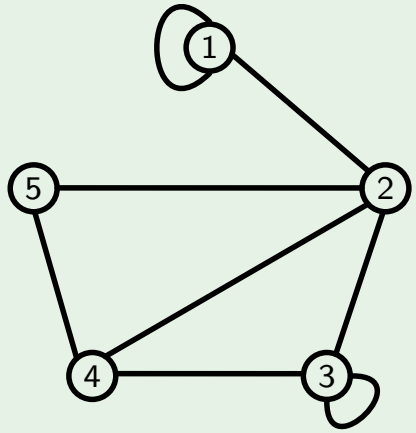
Definition

A digraph is a pair (V, A), where V is as before and A is a set of ordered pairs of elements of V.
That is, $A \subset \{(x, y) \colon x, y \in V\}$; A is called the arc set.

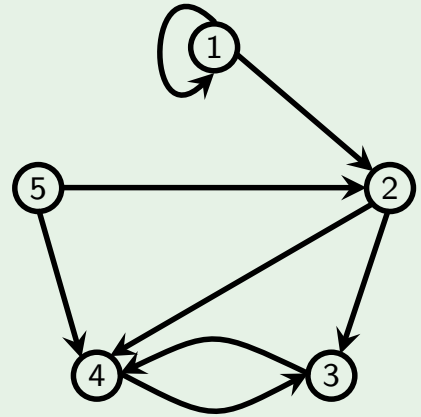
Example



multigraph

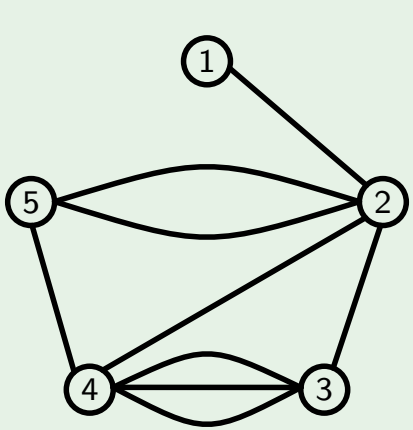


pseudograph

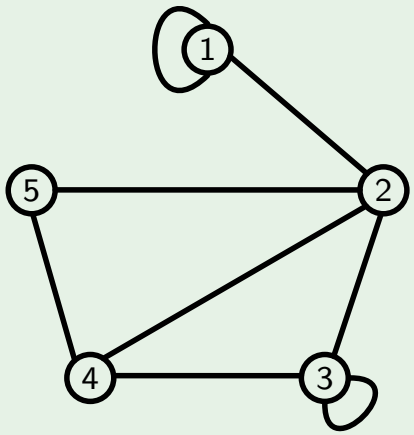


digraph

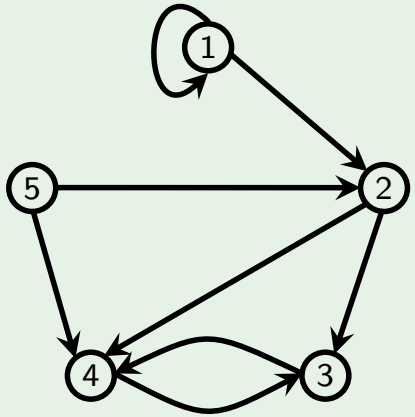
Example



multigraph



pseudograph



digraph

Remark: Unless stated otherwise, bfgraph means a simple graph.

Given a graph $G = (V, E)$

Definition

The degree of a vertex x in a graph G is the number of edges x belongs to.
Notation: $\deg_G(x)$ or $\mathrm{d}_G(x)$;
when there is no ambiguity, we write $\deg(x)$ or $\mathrm{d}(x)$.

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For the example on slide 3:
$\mathrm{d}(1) = 1$ – a pendant vertex; $\mathrm{d}(2) = 4$, $\mathrm{d}(3) = 2$, $\mathrm{d}(4) = 3$, $\mathrm{d}(5) = 2$.

If xy is an edge of G, the vertices x and y are adjacent, and both are incident with the edge $e = xy$.

Definition

The neighbourhood of a vertex x in a graph G is the set of all vertices of G adjacent to x.
Notation: $\mathrm{N}_G(x) = \mathrm{N}(x) := \{y \in V(G) \colon xy \in E(G)\}$

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Remark: In a simple graph $d(x) = |N(x)|$.

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For the example on slide 3:
$\mathrm{N}(3) = \{2, 4\}; \mathrm{N}(4) = \{2, 3, 5\}$

Theorem (Handshaking Lemma)

If $G = (V, E)$ is a simple graph, then

$$\sum_{v \in V} \mathrm{d}(v) = 2|E|.$$

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Corollary

In a simple graph, the number of vertices of odd degree is even.

Definition

The minimum degree of a graph G is the smallest degree among its vertices.

Notation: $\delta(G) := \min\{\mathrm{d}(x) \colon x \in V(G)\}$

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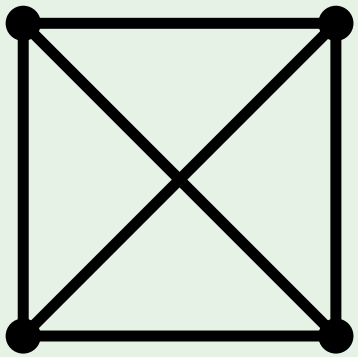
A graph is regular if all its vertices have the same degree. Then $\delta(G) = \Delta(G)$. If $\delta(G) = \Delta(G) = r$, the graph is called r-regular.

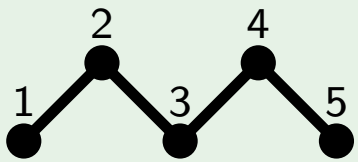
Important Classes of Graphs

- complete graph of order n – K_n
  $K_n = \left(V, \binom{V}{2}\right)$ – its edge set consists of **all** two-element subsets of V
- path of order n – P_n
  A graph whose vertices can be arranged in a sequence $(x_1, x_2, \ldots, x_n)$ such that $E(P_n) = \{x_i x_{i+1} : i \in \{1, 2, \ldots, n-1\}\}$.
- cycle of order n – C_n
  A graph whose vertices can be arranged in a sequence $(x_1, x_2, \ldots, x_n)$ such that $E(C_n) = \{x_i x_{i+1} : i \in \{1, 2, \ldots, n-1\}\} \cup \{x_1 x_n\}$.

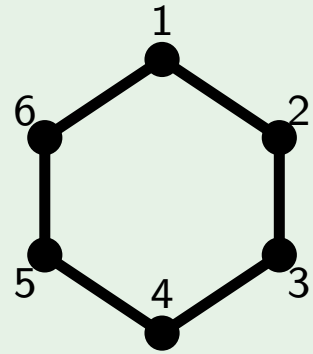
Example



complete graph K_4



path P_5



cycle C_6

Definition

A simple graph $G = (V, E)$ is bipartite if its vertex set can be partitioned into sets X and Y such that $V = X \cup Y$, $X \cap Y = \emptyset$, and each edge has exactly one endpoint in X and one in Y.

$xy \in E \Rightarrow x \in X \wedge y \in Y$

Notation: $G = (X, Y; E)$

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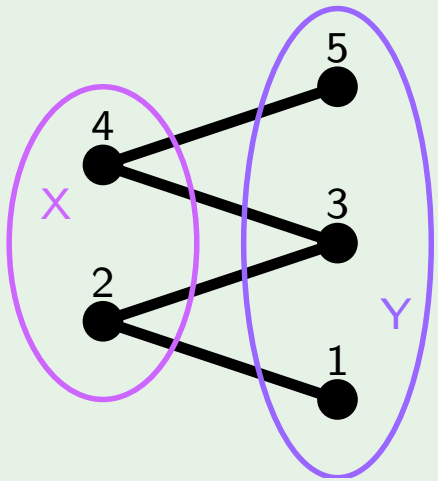
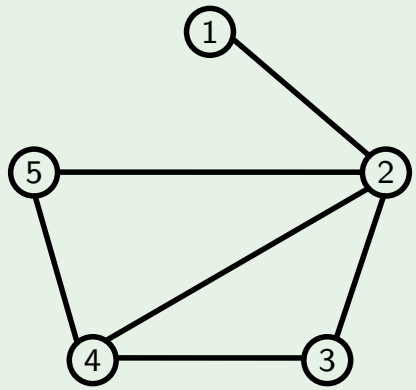
Definition

The complement of a graph $G = (V, E)$ has the same vertex set V, and its edge set is the complement of E in the edge set of the complete graph on V.
Notation: $\overline{G} := \left(V, \binom{V}{2} \setminus E\right)$

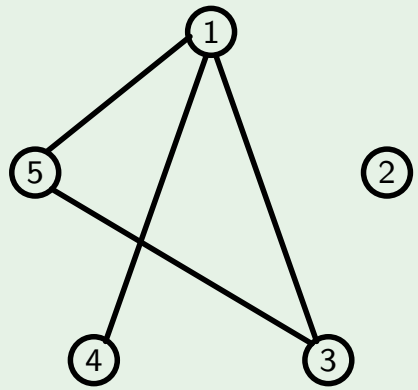
Example



bipartite graph



graph G



and its complement

Important Classes of Graphs (Continued)

- complete bipartite graph – $K_{p,q}$
  A bipartite graph with $|X| = p$, $|Y| = q$, whose edge set contains **every possible** pair with one vertex in X and one in Y,
  i.e. $E(K_{p,q}) = \{xy : x \in X \wedge y \in Y\}$

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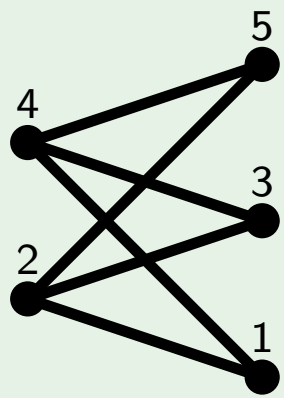

complete bipartite graph $K_{2,3}$

Graph Isomorphism

Definition

An isomorphism between graphs $G = (V, E)$ and $H = (W, F)$ is a bijection $f : V \to W$ such that $\forall x, y \in V : (xy \in E \Leftrightarrow f(x)f(y) \in F)$.

In other words, two vertices of G form an edge if and only if their images $f(x)$ and $f(y)$ form an edge in H.

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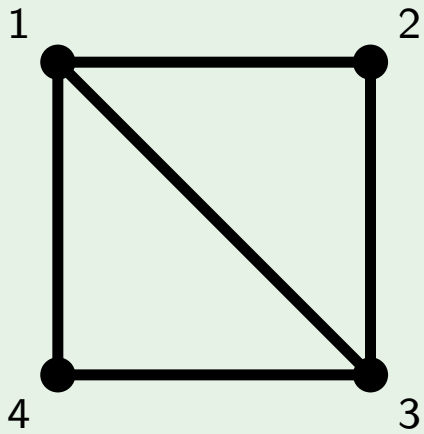
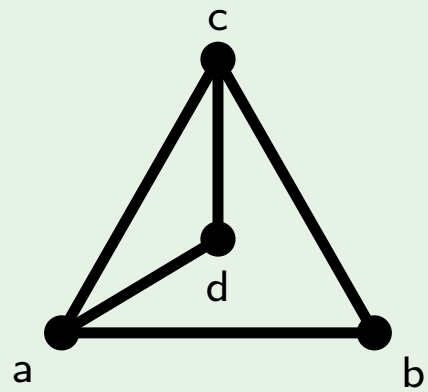
Informally, two graphs are isomorphic if they can be drawn in the same way, or if one can be obtained by moving the vertices of the other (with the edges moving along with their endpoints).

Example





graph $G = (V, E)$
$V = \{1, 2, 3, 4\}$
$E = \{12, 23, 34, 14, 13\}$

graph $H = (W, F)$
$W = \{a, b, c, d\}$
$F = \{ab, bc, cd, ac, ad\}$

The isomorphism $f : V \to W$ is given by:
$f(1) = a, f(2) = b, f(3) = c, f(4) = d$

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A graph $H = (W, F)$ is a subgraph of $G = (V, E)$ if $W \subset V$ and $F \subset E$.

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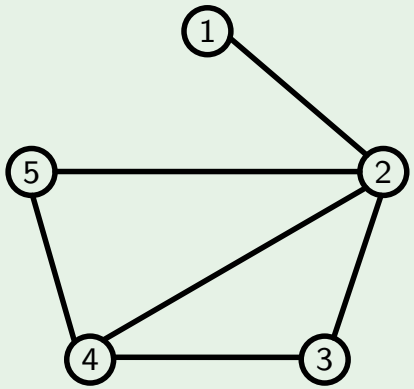
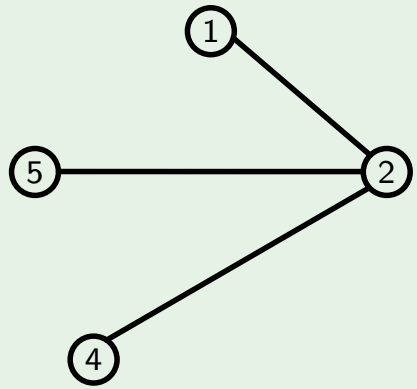
The edge set of the subgraph induced by S contains *all* edges of G *whose two endpoints belong to* S.

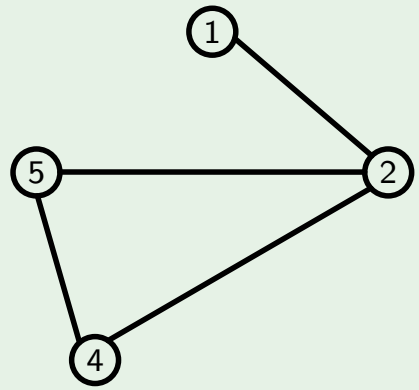
Example



graph G



subgraph of G



induced subgraph of G

Computer Representations of Graphs

Definition

The adjacency matrix of a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ is the square matrix $A(G) = [a_{ij}] \in M_{n \times n}$ such that

$$a_{ij} = \begin{cases} 1, & v_i v_j \in E(G) \\ 0, & v_i v_j \notin E(G) \end{cases}$$

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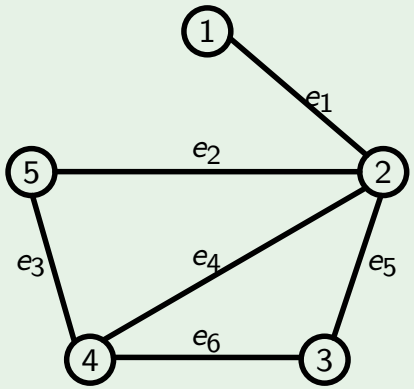
Definition

The incidence matrix of a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ is the matrix $B(G) = [b_{ij}] \in M_{n \times m}$ such that

$$b_{ij} = \begin{cases} 1, & v_i \in e_j \\ 0, & v_i \notin e_j \end{cases}$$

Example



$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad B(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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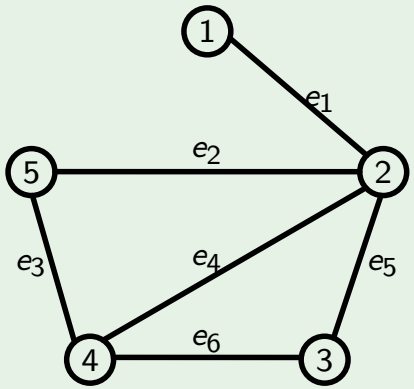

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Remarks:

The adjacency matrix is symmetric.

The degree of a vertex is the sum of the entries in the corresponding row of either matrix.

Every column of the incidence matrix contains exactly two ones.

A graph can also be represented in a computer using an adjacency list, which assigns to each vertex a list of its neighbours. These neighbours are usually listed in increasing order.

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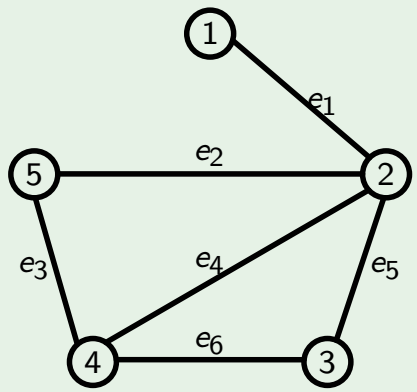

$$
\begin{aligned}
1 &\to 2 \\
2 &\to 1, 3, 4, 5 \\
3 &\to 2, 4 \\
4 &\to 2, 3, 5 \\
5 &\to 2, 4
\end{aligned}
$$